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Wigner functions for a class of semi-direct product groups

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Abstract

Following a general method proposed earlier, we construct here Wigner functions defined on coadjoint orbits of a class of semidirect product groups. The groups in question are such that their unitary duals consist purely of representations from the discrete series and each unitary irreducible representation is associated with a coadjoint orbit. The set of all coadjoint orbits (hence UIRs) is finite and their union is dense in the dual of the Lie algebra. The simple structure of the groups and the orbits enables us to compute the various quantities appearing in the definition of the Wigner function explicitly. A large number of examples, with potential use in image analysis, is worked out.

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1. Introduction

The Wigner function, a useful computational tool in atomic and quantum statistical physics, and more recently in quantum optics and image processing, was introduced by Wigner [1] as a *quasi-probability distribution* on phase space in the context of quasi-classical approximations. Using this distribution function, each quantum mechanical state can be represented as a real-valued (not necessarily positive) function of the position and momentum variables $\vec{q}$, $\vec{p}$, of a free particle moving on the classical (flat) phase space $\Gamma = \mathbb{R}^{2n}$. In view of the wide applicability of the Wigner function, and its deeper significance in harmonic analysis [2–5], it is useful to try to construct analogous (quasi-probability) distribution functions for more general phase spaces and quantum systems exhibiting specific symmetries. Such an attempt was initiated in some earlier work, where we developed a general procedure for constructing Wigner functions on coadjoint orbits of certain Lie groups [2, 6, 7]. As is well known from the Kirillov theory [8], coadjoint orbits of Lie groups carry symplectic structures, equipped with intrinsically defined Liouville-type invariant measures, making them resemble classical phase spaces. In particular, for semi-direct product groups of the type $G = \mathbb{R}^n \rtimes H$, where H is a closed subgroup of $GL(n, \mathbb{R})$, the coadjoint orbits of interest to us for the construction of Wigner functions, are cotangent bundles [9, 10], $T^*\hat{\mathcal{O}}$, of manifolds $\hat{\mathcal{O}}$ which are themselves orbits, under the action of H, of vectors in the the dual space $\hat{\mathbb{R}}^n$ (of the vector space $\mathbb{R}^n$). The construction of Wigner functions outlined in [2], and extended in [7], works particularly well for such groups and especially when the orbit $\hat{\mathcal{O}}$ is open and free. This means that these orbits have dimension n, are open sets of $\hat{\mathbb{R}}^n$ and the stability subgroup consists of just the identity element e of H. The unitary irreducible representations of these groups have the property of being *square integrable*, an aspect studied in detail in [11]. In this paper we shall explicitly construct Wigner functions for a number of such groups, which are of importance in image processing, signal analysis and quantum optics. We also derive expressions for the generic group $G = \mathbb{R}^n \rtimes H$. The advantage of working with such groups is that the various quantities which appear in the definition of the Wigner function can be explicitly calculated. For several of the examples worked out here, Wigner functions have also been constructed in the past using special techniques, specifically adapted to each group being examined [12, 13]. However, ours is a general technique, applicable to a wide class of Lie groups.

The rest of this paper is organized as follows: section 2 is a brief recapitulation of the standard Wigner function and its properties; section 3 lays out some preliminary mathematical properties of the kind of semidirect product groups, for which we shall be constructing Wigner functions in this paper. In section 4 we describe, in some detail, the symplectic structure of the coadjoint orbits of the groups in question and in section 5 we describe their associated unitary irreducible representations. Section 6 recapitulates the notion of a square-integrable representation and sets out some results on the square integrability of the irreducible representations introduced in the previous section. Sections 7, 8 and 9 lay down the main results of this paper, by first introducing the general construction of Wigner functions for square-integrable representations, followed by their properties and finally, the specialization of the construction to semidirect product groups with open free orbits. Section 10 is devoted to the question of the support of the Wigner functions constructed in the previous section. In section 11 a large number of explicit examples are worked out.

2. The standard Wigner function

Let us begin with a quick revision of some basic properties of the function defined in [1]. The quasiprobability distribution function W^{QM} is defined, on the flat phase space $\Gamma = \mathbb{R}^{2n}$, for any quantum mechanical state $\phi \in L^2(\mathbb{R}^n, \mathrm{d}\vec{x})$ as

$$W^{QM}(\phi|\vec{q}, \vec{p}; h) = \frac{1}{h^n} \int_{\mathbb{R}^n} \overline{\phi\left(\vec{q} - \frac{\vec{x}}{2}\right)}\, \mathrm{e}^{-\frac{2\pi \mathrm{i}\vec{x}\cdot\vec{p}}{h}} \phi\left(\vec{q} + \frac{\vec{x}}{2}\right) \mathrm{d}\vec{x}. \tag{1}$$

The variable $\vec{q}$ represents the position of the system, $\vec{p}$ its momentum at the point $\vec{q}$ and h is Planck's constant. The phase space Γ can also be viewed as an orbit $\mathcal{O}^*$ under the coadjoint action of the Heisenberg–Weyl group G_{HW} on the dual space $\mathfrak{g}^*_{HW}$ of its Lie algebra $\mathfrak{g}_{HW}$, and this is the point of departure in [2] for constructing a generalization of the Wigner function. Canonical transformations of the phase space Γ:

$$(\vec{q}, \vec{p}) \to (\vec{q} - \vec{q}_0, \vec{p} - \vec{p}_0) \tag{2}$$

which equivalently can be viewed as the coadjoint action of G_{HW} on $\mathfrak{g}^*_{HW}$, lead to unitary transformations on the space of wavefunctions ϕ:

$$\phi \longrightarrow U(\vec{q}_0, \vec{p}_0)\phi = e^{\frac{2\pi i}{h}(\vec{Q}\cdot\vec{p}_0 - \vec{P}\cdot\vec{q}_0)}\phi \tag{3}$$

where $\vec{Q}$ and $\vec{P}$ are the usual n-vector operators of position and momentum, respectively. It can be shown that the Wigner function satisfies the following covariance condition related to equations (2) and (3):

$$W^{QM}(U(\vec{q}_0, \vec{p}_0)\phi \,|\, \vec{q}, \vec{p}; h) = W^{QM}(\phi \,|\, \vec{q} - \vec{q}_0, \vec{p} - \vec{p}_0; h). \tag{4}$$

The other important property of the Wigner function is the existence of the marginality conditions, which make it resemble a classical probability distribution:

$$\int_{\mathbb{R}^{2n}} W^{QM}(\phi \,|\, \vec{q}, \vec{p}; h)\, \mathrm{d}\vec{p} = |\phi(\vec{q})|^2 \tag{5}$$

and

$$\int_{\mathbb{R}^{2n}} W^{QM}(\phi \,|\, \vec{q}, \vec{p}; h)\, \mathrm{d}\vec{q} = |\hat{\phi}(\vec{p})|^2 \tag{6}$$

where $\hat{\phi}$ is the Fourier transform of ϕ. However, as expected, for a given ϕ there exist in general regions of phase space over which the function $W^{QM}(\phi \,|\, \vec{q}, \vec{p}, h)$ can also assume negative values and hence W^{QM} cannot be a true probability density. An obvious generalization leads to a definition of the Wigner function for a pair of wavefunctions ϕ, ψ; one then has the well known *overlap condition*:

$$\int_{\mathbb{R}^2} \mathrm{d}\vec{q}\, \mathrm{d}\vec{p}\, W^{QM}(\phi, \psi|\vec{q}, \vec{p}) W^{QM}(v, w|\vec{q}, \vec{p}) = \langle\phi|w\rangle\langle v|\psi\rangle. \tag{7}$$

A general Wigner function, defined using some other group, should also preserve as many of the above properties as possible. Moreover, mathematically it should appear in much the same way, as a function defined on a coadjoint orbit. The work reported in [2] and [6] was an attempt to do this.

3. Some mathematical preliminaries

We begin with some basic properties of semi-direct product groups of the type mentioned above and in particular take a closer look at their non-trivial orbits, i.e. orbits of maximal dimension. Let $G = \mathbb{R}^n \rtimes H$ be the semidirect product group with elements $g = (\vec{b}, \mathbf{h})$, $\vec{b} \in \mathbb{R}^n$, and $\mathbf{h} \in H$ and the multiplication law:

$$(\vec{b}_1, \mathbf{h}_1)(\vec{b}_2, \mathbf{h}_2) = (\vec{b}_1 + \mathbf{h}_1\vec{b}_2, \mathbf{h}_1\mathbf{h}_2). \tag{8}$$

Here H is assumed to be a closed subgroup of $GL(n, \mathbb{R})$ and, as mentioned earlier, we will consider only the case where H is an n-dimensional subgroup of $GL(n, \mathbb{R})$ such that there exists at least one open free orbit, $\hat{\mathcal{O}}_{\vec{k}^T} = \{\vec{k}^T\mathbf{h} \,|\, \mathbf{h} \in H\}$, for some $\vec{k}^T$ in $\hat{\mathbb{R}}^n$ (the dual of $\mathbb{R}^n$). (Our convention is to use column vectors for elements of $\mathbb{R}^n$ and row vectors for those of $\hat{\mathbb{R}}^n$, the superscript T denoting a transpose. Also an $n \times n$ diagonal matrix, with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ will be denoted, $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n.)$ An element $g \in G$ can be written in matrix form as

$$g = \begin{pmatrix} \mathbf{h} & \vec{b} \\ \vec{0}^T & 1 \end{pmatrix}$$

where $\vec{0}^T$ is the zero vector in $\hat{\mathbb{R}}^n$. The inverse element is

$$g^{-1} = \begin{pmatrix} \mathbf{h}^{-1} & -\mathbf{h}^{-1}\vec{b} \\ \vec{0}^T & 1 \end{pmatrix}.$$

Note that $\mathbf{h}$ is an $n \times n$ matrix with non-zero determinant, which acts on $\vec{x} \in \mathbb{R}^n$ from the left in the usual way, $\vec{x} \mapsto \mathbf{h}\vec{x}$, and similarly, it acts on $\vec{x}^T \in \hat{\mathbb{R}}^n$ from the right, $\vec{x}^T \mapsto \vec{x}^T\mathbf{h}$. The left-invariant Haar measure, $\mathrm{d}\mu_G$, of G is

$$\mathrm{d}\mu_G(\vec{b}, \mathbf{h}) = \frac{1}{|\det \mathbf{h}|} \mathrm{d}\vec{b}\, \mathrm{d}\mu_H(\mathbf{h}) \tag{9}$$

$\mathrm{d}\vec{b}$ being the Lebesgue measure on $\mathbb{R}^n$ and $\mathrm{d}\mu_H$ the left-invariant Haar measure of H. While it is the left-invariant measure that we shall consistently use, it is nonetheless worthwhile to write down at this point the right-invariant Haar measure $\mathrm{d}\mu_r$ as well, in terms of the left Haar measure and the modular functions Δ_G, Δ_H, of the groups G and H, respectively:

$$\mathrm{d}\mu_G(\vec{b}, \mathbf{h}) = \Delta_G(\vec{b}, \mathbf{h})\, \mathrm{d}\mu_r(\vec{b}, \mathbf{h}) = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|} \mathrm{d}\mu_r(\vec{b}, \mathbf{h}). \tag{10}$$

Let $\mathfrak{g} = Lie(G)$ be the Lie algebra of G and $\{L^1, L^2, \ldots, L^{2n}\}$ a basis of it chosen so that the first n elements, $\{L^1, L^2, \ldots, L^n\}$, form a basis of $\mathfrak{h} = Lie(H)$, and the last n elements, $\{L^{n+1}, L^{n+2}, \ldots, L^{2n}\}$, which are the generators of translations, form a basis in $\mathbb{R}^n$. An element $X \in \mathfrak{g}$ can be written in matrix form as

$$X = x_1 L^1 + x_2 L^2 + \cdots + x_{2n} L^{2n} = \begin{pmatrix} X_q & \vec{x}_p \\ \vec{0}^T & 0 \end{pmatrix} \tag{11}$$

where X_q is an $n \times n$ matrix with entries depending on x_i, $i = 1, \ldots, n$, and $\vec{x}_p$ is a column vector with components $x_{n+1}, x_{n+2}, \ldots, x_{2n}$. Also, it will be useful to introduce the vector $\vec{x}_q$, with components x_i, $i = 1, \ldots, n$, and the vector of matrices $\mathfrak{X} = (L^1, L^2, \ldots, L^n)$. Next, for any $\vec{u} \in \mathbb{R}^n$, we define the matrix $[\mathfrak{X}\vec{u}]$ whose columns are the vectors $L^i\vec{u}$, $i = 1, 2, \ldots, n$,

$$[\mathfrak{X}\vec{u}] = [L^1\vec{u}, L^2\vec{u}, \ldots, L^n\vec{u}]. \tag{12}$$

The adjoint action of the group on its Lie algebra is given by

$$X \mapsto \mathrm{Ad}_g\, X := gXg^{-1} = \begin{pmatrix} \mathbf{h}X_q\mathbf{h}^{-1} & -\mathbf{h}X_q\mathbf{h}^{-1}\vec{b} + \mathbf{h}\vec{x}_p \\ \vec{0}^T & 0 \end{pmatrix}. \tag{13}$$

Introducing the matrix $M(\mathbf{h})$ such that

$$\mathbf{h}L^k\mathbf{h}^{-1} = \sum_{i=1}^{n} L^i M(\mathbf{h})_i^k \tag{14}$$

the adjoint action of an element $g = (\vec{b}, \mathbf{h}) \in G$ may conveniently be written in terms of its action on the $2n$-dimensional vector $\binom{\vec{x}_q}{\vec{x}_p}$ as

$$\begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix} \mapsto \begin{pmatrix} \vec{x}'_q \\ \vec{x}'_p \end{pmatrix} = M(\vec{b}, \mathbf{h}) \begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix} \tag{15}$$

where $M(\vec{b}, \mathbf{h})$ is the $2n \times 2n$-matrix

$$M(\vec{b}, \mathbf{h}) = \begin{pmatrix} M(\mathbf{h}) & \mathbb{O}_n \\ -[\mathfrak{X}\vec{b}]M(\mathbf{h}) & \mathbf{h} \end{pmatrix} \tag{16}$$

$\mathbb{O}_n$ being the $n \times n$ null matrix. Note that $M(\mathbf{h})$ is just the matrix of the adjoint action of $\mathbf{h}$ on $\mathfrak{h}$ (the Lie algebra of H) computed with respect to the basis $\{L^1, L^2, \ldots, L^n\}$. Similarly, $M(\vec{b}, \mathbf{h})$ is the matrix of the adjoint action of $g = (\vec{b}, \mathbf{h})$ on $\mathfrak{g}$, the Lie algebra of G. By abuse of notation, we shall also write

$$\mathrm{Ad}_h\, X_q = M(\mathbf{h})\vec{x}_q \qquad \mathrm{Ad}_g\, X = M(\vec{b}, \mathbf{h}) \begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix}. \tag{17}$$

The coadjoint action of G on $\mathfrak{g}^*$, the dual space of its Lie algebra, can now be immediately read off from equation (16). Indeed, let $\{L_i^*\}_{i=1}^{2n}$ be the basis of $\mathfrak{g}^*$ which is dual to the basis $\{L^i\}_{i=1}^{2n}$ of $\mathfrak{g}$, i.e.,

$$\langle L_i^*; L^j\rangle = \delta_i^j \qquad i, j = 1, 2, \ldots, 2n.$$

A general element $X^* \in \mathfrak{g}^*$ then has the form

$$X^* = \sum_{i=1}^{n} \gamma^i L_i^* \qquad \gamma^i \in \mathbb{R}$$

and again we introduce the row vectors

$$\vec{\gamma}^T = (\gamma^1, \gamma^2, \ldots, \gamma^{2n}) \qquad \vec{\gamma}_q^T = (\gamma^1, \gamma^2, \ldots, \gamma^n) \qquad \vec{\gamma}_p^T = (\gamma^{n+1}, \gamma^{n+2}, \ldots, \gamma^{2n}).$$

Using the relation

$$\langle \mathrm{Ad}_g^\sharp X^*; X\rangle = \langle X^*; \mathrm{Ad}_{g^{-1}} X\rangle$$

we easily obtain from equation (16) and with the same abuse of notation as in equation (17),

$$\mathrm{Ad}^\sharp_{(\vec{b},\mathbf{h})} X^* = \left(\vec{\gamma}_q^T, \vec{\gamma}_p^T\right) M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) = \left(\vec{\gamma}_q^T, \vec{\gamma}_p^T\right)\begin{pmatrix} M(\mathbf{h}^{-1}) & \mathbb{O}_n \\ \mathbf{h}^{-1}[\mathfrak{X}\vec{b}] & \mathbf{h}^{-1}\end{pmatrix}. \tag{18}$$

Thus, under the coadjoint action, a vector $\left(\vec{\gamma}_q^T, \vec{\gamma}_p^T\right)$ changes to

$$\begin{aligned} \vec{\gamma}_q^{\prime T} &= \vec{\gamma}_q^T M(\mathbf{h}^{-1}) + \vec{\gamma}_p^T \mathbf{h}^{-1}[\mathfrak{X}\vec{b}] \\ \vec{\gamma}_p^{\prime T} &= \vec{\gamma}_p^T \mathbf{h}^{-1}. \end{aligned} \tag{19}$$

Let us also note that the modular functions appearing in equation (10) can be written [14] in terms of the coadjoint operators as

$$\Delta_G(\vec{b}, \mathbf{h}) = \left|\det \mathrm{Ad}^\sharp_{(\vec{b},\mathbf{h})}\right| = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|} \qquad \Delta_H(\mathbf{h}) = \left|\det \mathrm{Ad}^\sharp_{\mathbf{h}}\right| = \frac{1}{|\det M(\mathbf{h})|}. \tag{20}$$

Before leaving this section we make a further important assumption on the nature of the group G. We require that the range in G of the exponential map be a dense set whose complement has Haar measure zero. (This includes, for example, groups of exponential type.) Thus, by exponentiating equation (11), we may write any element (up to a set of measure zero) of G as

$$g = \mathrm{e}^X = \begin{pmatrix} \mathrm{e}^{X_q} & \mathrm{e}^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2} \vec{x}_p \\ \vec{0}^T & 1 \end{pmatrix} \qquad X \in N \tag{21}$$

where $N \in \mathfrak{g}$ is the domain of the exponential map, which contains the origin and has the property that if $X \in N$ then $-X \in N$. The $n \times n$ matrix sinch A is defined as the sum of an infinite series:

$$\operatorname{sinch} A = \mathbb{I}_n + \frac{1}{3!}A^2 + \frac{1}{5!}A^4 + \frac{1}{7!}A^6 + \cdots \tag{22}$$

$\mathbb{I}_n$ being the $n \times n$ unit matrix. When the matrix A has an inverse, sinch A can also be formally written as

$$\operatorname{sinch} A = \frac{\mathrm{e}^A - \mathrm{e}^{-A}}{2A} = A^{-1}\sinh A. \tag{23}$$

It will also be useful to introduce the matrix valued functions

$$F(X_q) = \mathrm{e}^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2} = \mathbb{I}_n + \frac{X_q}{2!} + \frac{X_q^2}{3!} + \frac{X_q^3}{4!} + \cdots \tag{24}$$

and

$$F(-X_q)^{-1} = \mathrm{e}^{X_q} F(X_q)^{-1} = \frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}} = \mathbb{I}_n + \frac{X_q}{2} + \sum_{k\geqslant 1} (-1)^{k-1} \frac{B_k X_q^{2k}}{(2k)!} \tag{25}$$

where the B_k are the Bernoulli numbers, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc, and generally

$$B_k = \frac{(2k)!}{\pi^{2k} 2^{2k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \qquad k = 1, 2, 3, \ldots.$$

Later we shall need to express the Haar measure $\mathrm{d}\mu_G$ in terms of the coordinates of the Lie algebra, using the exponential map. Writing

$$\mathrm{d}\mu_G(\mathrm{e}^X) = m_G(\vec{x}_q, \vec{x}_p)\, \mathrm{d}\vec{x}_q\, \mathrm{d}\vec{x}_p \tag{26}$$

the density function $m_G(\vec{x}_q, \vec{x}_p)$ is easily calculated, using equation (21). Indeed, from equation (9),

$$\begin{aligned} \mathrm{d}\mu_G(e^X) &= \mathrm{d}\mu_G\left(\mathrm{e}^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2} \vec{x}_p, \mathrm{e}^{X_q}\right) \\ &= \frac{1}{|\det \mathrm{e}^{X_q}|} \left|\det\left(\mathrm{e}^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2}\right)\right| \mathrm{d}\mu_H(\mathrm{e}^{X_q})\, \mathrm{d}\vec{x}_p \\ &= \left|\det\left(\mathrm{e}^{\frac{-X_q}{2}} \operatorname{sinch} \frac{X_q}{2}\right)\right| \mathrm{d}\mu_H(\mathrm{e}^{X_q})\, \mathrm{d}\vec{x}_p \end{aligned} \tag{27}$$

It is also possible to write $\mathrm{d}\mu_H(\mathrm{e}^{X_q})$ in terms of the Lebesgue measure, $\mathrm{d}\vec{x}_q = \mathrm{d}x_1\, \mathrm{d}x_2, \ldots, \mathrm{d}x_n$, times some density function m_H [15],

$$\begin{aligned} \mathrm{d}\mu_H(\mathrm{e}^{X_q}) &= m_H(\vec{x}_q)\, \mathrm{d}\vec{x}_q \\ &= \left|\det \frac{1 - \mathrm{e}^{-a\,\mathrm{d}X_q}}{a\,\mathrm{d}X_q}\right| \mathrm{d}\vec{x}_q = \left|\det\left(\mathrm{e}^{-a\,\mathrm{d}\frac{X_q}{2}} \operatorname{sinch}\left(a\,\mathrm{d}\frac{X_q}{2}\right)\right)\right| \mathrm{d}\vec{x}_q \end{aligned} \tag{28}$$

where $a\,\mathrm{d}X$ is the linear map on $\mathfrak{g}$ which is the infinitesimal generator of the adjoint action Ad_g, $g \in G$:

$$a\,\mathrm{d}X(L) = [X, L] \qquad \text{and} \qquad \mathrm{Ad}_g = \mathrm{Ad}_{(\mathrm{e}^X)} = \mathrm{e}^{a\,\mathrm{d}X}. \tag{29}$$

Finally, the left Haar measure on G takes the form:

$$\mathrm{d}\mu_G(\mathrm{e}^X) = \left|\det\left(\mathrm{e}^{\frac{-X_q}{2}} \operatorname{sinch}\left(\frac{X_q}{2}\right)\right) \det\left(\mathrm{e}^{-a\,\mathrm{d}\frac{X_q}{2}} \operatorname{sinch}\left(a\,\mathrm{d}\frac{X_q}{2}\right)\right)\right| \mathrm{d}\vec{x}_q\, \mathrm{d}\vec{x}_p \tag{30}$$

and the density function $m_G(\vec{x})$ appearing in equation (26) is

$$\begin{aligned} m_G(\vec{x}) &= \left|\det\left(\mathrm{e}^{-\frac{X_q}{2}} \operatorname{sinch}\left(\frac{X_q}{2}\right)\right) \det\left(\mathrm{e}^{-a\,\mathrm{d}\frac{X_q}{2}} \operatorname{sinch}\left(a\,\mathrm{d}\frac{X_q}{2}\right)\right)\right| \\ &= |\det F(-X_q) \det F(-a\,\mathrm{d}X_q)| \end{aligned} \tag{31}$$

4. Orbits and invariant measures

It is now possible to determine the non-trivial coadjoint orbits of G, which will be the main focus of our attention. These are orbits of fixed vectors in $\mathfrak{g}^*$ under the coadjoint action in equation (19). Consider first the vector $(\vec{0}^T, \vec{k}^T) \in \mathbb{R}^{2n}$, $\vec{k} \neq \vec{0}$ and let $\mathcal{O}^*_{(\vec{0}^T, \vec{k}^T)}$ be its orbit under the coadjoint action, i.e.,

$$\mathcal{O}^*_{(\vec{0}^T, \vec{k}^T)} = \left\{\left(\vec{\gamma}_q^T, \vec{\gamma}_p^T\right) = (\vec{0}^T, \vec{k}^T) M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) \mid (\vec{b}, \mathbf{h}) \in \mathbb{R}^n \rtimes H\right\}. \tag{32}$$

Then, from equation (19),

$$\begin{aligned} \vec{\gamma}_q^T &= \vec{k}^T\mathbf{h}^{-1}[\boldsymbol{\mathfrak{X}}\vec{b}] = \vec{\gamma}_p^T[\boldsymbol{\mathfrak{X}}\vec{b}] \\ \vec{\gamma}_p^T &= \vec{k}^T\mathbf{h}^{-1}. \end{aligned} \tag{33}$$

The vectors $\vec{\gamma}_p^T$ generate the orbit $\hat{\mathcal{O}}_{\vec{k}^T}$ of the subgroup H in $\hat{\mathbb{R}}$. We now show that, for any $\vec{\gamma}_p^T$, the vector $\vec{\gamma}_q^T = \vec{\gamma}_p^T[\boldsymbol{\mathfrak{X}}\vec{b}]$ can be identified with an element of the cotangent space of $\hat{\mathcal{O}}_{\vec{k}^T}$ at this point. Indeed, for any $i = 1, 2, \ldots, n$, consider a curve, $\vec{u}^i(t)^T$ in $\hat{\mathcal{O}}_{\vec{k}^T}$ of the type

$$\vec{u}^i(t)^T = \vec{\gamma}_p^T \mathrm{e}^{L^i t} \qquad t \in [-\epsilon, \epsilon] \subset \mathbb{R}. \tag{34}$$

Then, $\vec{u}^i(0)^T = \vec{\gamma}_p^T$, and

$$\left.\frac{\mathrm{d}\vec{u}^i(t)^T}{\mathrm{d}t}\right|_{t=0} = \vec{\gamma}_p^T L^i := \vec{t}_p^{iT} \tag{35}$$

is a vector tangent to $\hat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Recall that we are assuming that the action of H on $\hat{\mathbb{R}}$ is open free. Hence the stability subgroup of the vector $\vec{k}^T$ under the action $\vec{k}^T \mapsto \vec{k}^T\mathbf{h}^{-1}$ is just the unit element of H and the orbit $\hat{\mathcal{O}}_{\vec{k}^T}$ is an open set of $\hat{\mathbb{R}}^n$, consequently of dimension n. This implies that the vectors $\vec{t}_p^{iT}$ are non-zero and linearly independent and hence form a basis for the tangent space $T_{\vec{\gamma}_p^T}\hat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Let $\vec{t}_p^{iT} = (t^{i1}, t^{i2}, \ldots, t^{in})$, in components, and define the matrix

$$\mathbf{T}(\vec{\gamma}_p^T) = [t^{ij}] = [\vec{\theta}_1, \vec{\theta}_2, \ldots, \vec{\theta}_n] \tag{36}$$

where the vectors $\vec{\theta}_i$ are its columns, with elements t^{ji}, $j = 1, 2, \ldots, n$. The vectors $\vec{\theta}_i$ form a basis for the cotangent space $T^*_{\vec{\gamma}_p^T}\hat{\mathcal{O}}_{\vec{k}^T}$ of $\hat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Thus, if b^i are the components of the vector $\vec{b}$, we have

$$\vec{\gamma}_p^T[\boldsymbol{\mathfrak{X}}\vec{b}] = \sum_{i=1}^n b^i\vec{\theta}_i^T \tag{37}$$

implying that $\left[\vec{\gamma}_p^T[\boldsymbol{\mathfrak{X}}\vec{b}]\right]^T = [\boldsymbol{\mathfrak{X}}\vec{b}]^T\vec{\gamma}_p$ is just a cotangent vector at $\vec{\gamma}_p^T$. Letting $\vec{b}$ run through all of $\mathbb{R}^n$, these vectors generate the whole cotangent space at $\vec{\gamma}_p^T$. Thus,

$$\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)} = T^*\hat{\mathcal{O}}_{\vec{k}^T} \tag{38}$$

and if $\vec{k}^T$ is a vector such that the orbit $\hat{\mathcal{O}}_{\vec{k}^T}$ is open and free, the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}$ has dimension $2n$. It is known [11] that if one such open free orbit exists, then there exists a finite discrete set of them, corresponding to vectors $\vec{k}_j^T$, $j = 1, 2, \ldots, N < \infty$, for which $\cup_{j=1}^N\hat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in $\hat{\mathbb{R}}^n$.

Similarly, let us compute the coadjoint orbit of a vector $(\vec{x}^T, \vec{0}^T) \in \hat{\mathbb{R}}^{2n}$. As before

$$\mathcal{O}^*_{(\vec{x}^T,\vec{0}^T)} = \left\{\left(\vec{\gamma}_q^T, \vec{\gamma}_p^T\right) = (\vec{x}^T, \vec{0}^T)M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) | (\vec{b}, \mathbf{h}) \in \mathbb{R}^n \rtimes H\right\} \tag{39}$$

and again from equation (19),

$$\begin{aligned} \vec{\gamma}_q^T &= \vec{x}^T M(\mathbf{h}^{-1}) \\ \vec{\gamma}_p^T &= \vec{0}^T \end{aligned} \tag{40}$$

and these orbits all have dimension lower than $2n$. From the point of view of the representation theory, these are the trivial orbits. Using the coordinates γ^i to identify $\mathbf{g}^*$ with $\hat{\mathbb{R}}^{2n}$, we arrive at the result:

Theorem 4.1. *If the action of H on $\hat{\mathbb{R}}^n$ is open free, the set of non-trivial coadjoint orbits in $\mathfrak{g}^*$ is finite and discrete and their union is dense in $\mathfrak{g}^*$. Moreover, each non-trivial coadjoint orbit, $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, is the cotangent bundle, $T^*\hat{\mathcal{O}}_{\vec{k}_j^T}$, of an open free orbit, $\hat{\mathcal{O}}_{\vec{k}_j^T} \subset \hat{\mathbb{R}}^n$, of a vector $\vec{k}_j^T \in \hat{\mathbb{R}}^n$ under the action of H. Under the coadjoint action of $G = \mathbb{R}^n \rtimes H$, the dual space of its Lie algebra decomposes as*

$$\mathfrak{g}^* \simeq \hat{\mathbb{R}}^{2n} = \left[\cup_{j=1}^N \mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}\right] \cup V = \left[\cup_{j=1}^N T^*\hat{\mathcal{O}}_{\vec{k}_j^T}\right] \cup V \tag{41}$$

where V is a set consisting of lower (than $2n$) dimensional orbits and therefore of Lebesgue measure zero in $\hat{\mathbb{R}}^{2n}$.

The orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, being *homogeneous symplectic manifolds* [8], carry invariant measures under the coadjoint action of equations (18) and (19). Indeed, if $\mathrm{d}\vec{\gamma}^T$ denotes the Lebesgue measure $\mathrm{d}\gamma^1\,\mathrm{d}\gamma^2, \ldots, \mathrm{d}\gamma^{2n}$, restricted to the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, then using equations (19) and (20) it is easy to check that under the coadjoint action it transforms as

$$\mathrm{d}\vec{\gamma}'^T = \Delta_G(\vec{b}, \mathbf{h})\,\mathrm{d}\vec{\gamma}^T. \tag{42}$$

On the other hand, the mapping $\kappa_j : \hat{\mathcal{O}}_{\vec{k}_j^T} \rightarrow H$,

$$\kappa_j(\vec{\gamma}_p^T) = \mathbf{h} \qquad \text{where} \quad \vec{\gamma}_p^T = \vec{k}_j^T\mathbf{h}^{-1} \tag{43}$$

is a homeomorphism. It is then straightforward to see that the measure

$$\mathrm{d}\Omega_j\big(\vec{\gamma}_q^T, \vec{\gamma}_p^T\big) = \sigma_j(\vec{\gamma}^T)^{-1}\,\mathrm{d}\vec{\gamma}^T \qquad \sigma_j(\vec{\gamma}^T) = \frac{\Delta_H\big[\kappa_j\big(\vec{\gamma}_p^T\big)\big]}{\big|\det\big[\kappa_j\big(\vec{\gamma}_p^T\big)\big]\big|} \tag{44}$$

is invariant on $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ under the coadjoint action.

Note, finally, that each one of the orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ is homeomorphic to the group G itself. Indeed, using equations (33), (37) and (43) let us define a map,

$$\tilde{\kappa}_j : \mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)} \longrightarrow \mathbb{R}^n \rtimes H \qquad \tilde{\kappa}_j\big(\vec{\gamma}_q^T, \vec{\gamma}_p^T\big) = (\vec{b}, \mathbf{h}) = \big(\mathbf{T}\big(\vec{\gamma}_p^T\big)^{-1}\vec{\gamma}_q, \kappa_j\big(\vec{\gamma}_p^T\big)\big) \tag{45}$$

where $\mathbf{T}\big(\vec{\gamma}_p^T\big)$ is the matrix of tangent vectors defined in equation (36). Then, $\tilde{\kappa}_j$ is a homeomorphism and it is straightforward to verify that

$$\tilde{\kappa}_j \circ \mathrm{Ad}^\sharp_{g_0} = L_{g_0} \circ \tilde{\kappa}_j \tag{46}$$

where $L_{g_0}(g) = g_0 g$, $g \in G$. More explicitly, if $\big(\vec{\gamma}_q^T, \vec{\gamma}_p^T\big) \mapsto g = (\vec{b}, \mathbf{h})$ under $\tilde{\kappa}_j$, then

$$\mathrm{Ad}^\sharp_{g_0}\big(\vec{\gamma}_q^T, \vec{\gamma}_p^T\big) \longmapsto (\vec{b}_0, \mathbf{h}_0)(\vec{b}, \mathbf{h}) = (\vec{b}_0 + \mathbf{h}\vec{b}, \mathbf{h}_0\mathbf{h}). \tag{47}$$

In other words, the homeomorphism $\tilde{\kappa}_j$, from the coadjoint orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ to the group $\mathbb{R}^n \rtimes H$, intertwines the coadjoint action on the orbit with the left action on the group and furthermore, under this homeomorphism the invariant measure $\mathrm{d}\Omega_j$ on the orbit transforms to the left Haar measure $\mathrm{d}\mu_G$ on the group.

Before leaving this section, we describe a second, in a way more intrinsic, method for arriving at the invariant measure in equation (44), using the fact that the orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ are symplectic manifolds and thus carry G-invariant two-forms [8] which can be computed using the structure constants of the group. As before, $\{L^i\}_{i=1}^{2n}$ will be a basis for the Lie algebra $\mathfrak{g}$ and $\{L_i^*\}_{i=1}^{2n}$ the dual basis of $\mathfrak{g}^*$. The Lie algebra of the group G is determined by the commutation relations

$$[L^i, L^j] = \sum_{k=1}^{2n} c_k^{ij} L^k \tag{48}$$

where the c_k^{ij} are the structure constants. Thus, in this basis, the linear map $a\,\mathrm{d}L^i$ has the matrix elements $[a\,\mathrm{d}L^i]_k^j = c_k^{ij}$. Let $X^* = \sum_{i=1}^{2n} \gamma^i L_i^* \in \mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)} \subset \mathfrak{g}^*$ and let us define a matrix $\Theta(\vec{\gamma}^T)$ at this point by

$$[\Theta(\vec{\gamma}^T)]^{ij} = \sum_{k=1}^{n} [a\,\mathrm{d}L^i]_k^j \gamma^k = \sum_{k=1}^{n} c_k^{ij} \gamma^k. \tag{49}$$

Using the $\Theta(\vec{\gamma}^T)$ matrix, we can now identify the Lie algebra $\mathfrak{g}$ with the tangent space, $T_{X^*}\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, to the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ at the point X^*. (Note that this tangent space is naturally isomorphic to $\mathfrak{g}^*$ itself.) Since the orbit $\hat{\mathcal{O}}_{\vec{k}_j^T}$ is open free, it has dimension n and its cotangent bundle, i.e. the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, has dimension $2n$. Thus, $T_{X^*}\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ has dimension $2n$ and in it we shall use the standard basis $\left\{\frac{\partial}{\partial\gamma^i}\right\}_{i=1}^{2n}$. Similarly, we shall use the dual basis $\{\mathrm{d}\gamma^i\}_{i=1}^{2n}$ for the cotangent space, $T^*_{X^*}\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$. From equation (49) we see that for $i = 1, 2, \ldots, 2n$, the vectors $\sum_{j=1}^{2n} [\Theta(\vec{\gamma}^T)]^{ij} \frac{\partial}{\partial\gamma^j}$ form a linearly independent set of tangent vectors to the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ at the point X^* (under the coadjoint action). Thus, for $X = \sum_{i=1}^{2n} x_i L^i \in \mathfrak{g}$, it follows that $\sum_{j=1}^{n} [\Theta(\vec{\gamma}^T)\vec{x}]^j \frac{\partial}{\partial\gamma^j}$ defines a vector in $T_{X^*}\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ and hence we have the identification map, $\phi_{X^*} : \mathfrak{g} \longrightarrow T_{X^*}\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$,

$$\phi_{X^*}(X) = \sum_{i,j,k=1}^{2n} c_k^{ij} x_i \gamma^k \frac{\partial}{\partial\gamma^j} = \sum_{i,j=1}^{2n} [\Theta(\vec{\gamma}^T)]^{ij} x_i \frac{\partial}{\partial\gamma^j} \tag{50}$$

as an isomorphism of vector spaces. The G-invariant 2-form (symplectic form) is then defined as

$$\omega_{X^*}(\phi_{X^*}(X), \phi_{X^*}(L)) = \langle X^*; [X, L]\rangle \tag{51}$$

which using equations (48) and (50) can be expressed in the form

$$\omega_{X^*} = \sum_{i,j=1}^{2n} [\omega_{X^*}]_{ij}\, \mathrm{d}\gamma^i \wedge \mathrm{d}\gamma^j = \sum_{i,j=1}^{2n} [\Theta(\vec{\gamma}^T)]_{ij}\, \mathrm{d}\gamma^j \wedge \mathrm{d}\gamma^i \tag{52}$$

where $[\Theta(\vec{\gamma}^T)]_{ij}$ are the elements of the inverse matrix $[\Theta(\vec{\gamma}^T)]^{-1}$. From this the G-invariant measure on the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ is computed to be

$$\mathrm{d}\Omega_j(\vec{\gamma}^T) = \lambda(\det[\omega_{X^*}])^{\frac{1}{2}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2, \ldots, \mathrm{d}\gamma^{2n} = \frac{\lambda}{(\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2, \ldots, \mathrm{d}\gamma^{2n}$$

where λ is a constant. By multiplying the basis vectors L^i by appropriate constants, λ can be made equal to one. We shall assume that this has been done and then write

$$\mathrm{d}\Omega_j(\vec{\gamma}^T) = (\det[\omega_{X^*}])^{\frac{1}{2}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2, \ldots, \mathrm{d}\gamma^{2n} = \frac{1}{(\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2, \ldots, \mathrm{d}\gamma^{2n}. \tag{53}$$

Comparing with equation (44) we find

$$\sigma_j(\vec{\gamma}^T) = (\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}. \tag{54}$$

5. Representations of *G*

In order to construct Wigner functions for the group $G = \mathbb{R}^n \rtimes H$ we shall use its quasi-regular representation. This representation acts via the unitary operators $U(\vec{b}, \mathbf{h})$ on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^n, \mathrm{d}\vec{x})$:

$$(U(\vec{b}, \mathbf{h})f)(\vec{x}) = |\det \mathbf{h}|^{-\frac{1}{2}} f(\mathbf{h}^{-1}(\vec{x} - \vec{b})) \qquad f \in \mathfrak{H}. \tag{55}$$

This representation is in general not irreducible, but is always multiplicity free. Moreover, the existence of open free orbits implies that every non-trivial irreducible sub-representation of G is contained in U and each such representation is *square integrable* [11] in a sense to be made precise presently.

In order to obtain the irreducible sub-representations of U, it is useful to look at the unitarily equivalent representation $\hat{U}(\vec{b}, \mathbf{h}) = \mathcal{F}U(\vec{b}, \mathbf{h})\mathcal{F}^{-1}$, where $\mathcal{F} : L^2(\mathbb{R}^n, \mathrm{d}\vec{x}) \to L^2(\hat{\mathbb{R}}^n, \mathrm{d}\vec{k}^T)$ is the Fourier transform operator:

$$(\mathcal{F}f)(\vec{k}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\vec{k}^T\vec{x}} f(\vec{x})\, \mathrm{d}\vec{x}.$$

The action of $\hat{U}(\vec{b}, \mathbf{h})$ on a vector $\hat{f} \in \hat{\mathfrak{H}} = L^2(\hat{\mathbb{R}}^n, \mathrm{d}\vec{k}^T)$ is easily seen to have the form

$$(\hat{U}(\vec{b}, \mathbf{h})\hat{f})(\vec{k}^T) = |\det \mathbf{h}|^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\vec{k}^T\vec{b}} \hat{f}(\vec{k}^T\mathbf{h}). \tag{56}$$

We shall also need the form of this representation, written in terms of Lie algebra variables, using the exponential map in equation (21):

$$(\hat{U}(e^{-X})\hat{f})(\vec{k}^T) = |\det[\mathrm{e}^{-X_q}]|^{\frac{1}{2}} \mathrm{e}^{(\mathrm{i}\vec{k}^T F(-X_q)\vec{x}_p)} \hat{f}(\vec{k}^T \mathrm{e}^{-X_q}). \tag{57}$$

Let $\vec{k}_j^T \in \hat{\mathbb{R}}^n$, $j = 1, 2, \ldots, N$, be a maximal set of vectors whose orbits $\hat{\mathcal{O}}_{\vec{k}_j^T}$ under H are open free and mutually disjoint. Then by theorem 4.1, $\cup_{j=1}^N \hat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in $\hat{\mathbb{R}}^n$ and $\cup_{j=1}^N T^*\hat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in the dual, $\mathfrak{g}^*$, of the Lie algebra of G. Set $\hat{\mathfrak{H}}_j = L^2(\hat{\mathcal{O}}_{\vec{k}_j^T}, \mathrm{d}\vec{k}^T)$ (the restriction of the Lebesgue measure to the orbit is implied). Then, it is not hard to see that each of these spaces is an invariant subspace for $\hat{U}$. Moreover, the restriction $\hat{U}_j$, of $\hat{U}$ to $\hat{\mathfrak{H}}_j$, is irreducible [11], and is in fact the representation of $\mathbb{R}^n \rtimes H$ which is induced from the character $\chi_j(\vec{x}) = \exp(\mathrm{i}\vec{k}_j^T\vec{x})$ of the Abelian subgroup $\mathbb{R}^n$. Thus,

$$\hat{\mathfrak{H}} = \oplus_{j=1}^N \hat{\mathfrak{H}}_j \qquad \hat{U}(\vec{b}, \mathbf{h}) = \oplus_{j=1}^N \hat{U}_j(\vec{b}, \mathbf{h}) \tag{58}$$

and it follows from Mackey's theory of induced representations [16] for semidirect product groups that these irreducible representations exhaust all non-trivial irreducible representations of G.

6. Square integrability of representations

The irreducible representations $\hat{U}_j$ in equation (58) all have one other property, of importance to us here. These representations are *square integrable* [9]. Recall that a unitary irreducible representation U of a group G on a Hilbert space $\mathfrak{H}$ is square integrable if there exists a non-zero vector $\eta \in \mathfrak{H}$, called an *admissible vector*, such that

$$c(\eta) = \int_G |\langle U(g)\eta|\eta\rangle|^2\, \mathrm{d}\mu_G(g) < \infty. \tag{59}$$

The existence of one such vector and irreducibility of the representation imply that the set of all admissible vectors $\mathcal{A}$ is dense in $\mathfrak{H}$. If the group is unimodular then $\mathcal{A}$ coincides with $\mathfrak{H}$, otherwise it is a proper subset of it. (For a more detailed description of square integrable representations and their properties, see e.g. [9]). For any square integrable representation U there exists a unique positive operator C on $\mathfrak{H}$ whose domain coincides with $\mathcal{A}$ and such that if $\eta_1, \eta_2 \in \mathcal{A}$ and $\phi_1, \phi_2 \in \mathfrak{H}$ the following *orthogonality relation* holds:

$$\int_G \overline{\langle U(g)\eta_2|\phi_2\rangle}\langle U(g)\eta_1|\phi_1\rangle \, \mathrm{d}\mu_G = \langle C\eta_1|C\eta_2\rangle\langle\phi_2|\phi_1\rangle. \tag{60}$$

This result is due to Duflo and Moore [17] and the operator C is usually referred to in the literature as the Duflo–Moore operator. If G is unimodular, C is a multiple of the identity, otherwise, it is an unbounded operator.

For semidirect product groups $G = \mathbb{R}^n \rtimes H$ of the type discussed in the previous sections, with open free orbits, the irreducible representations $\hat{U}_j(\vec{b}, \mathbf{h})$, appearing in the decomposition of equation (58) are all square integrable and one has the result [11]:

Theorem 6.1. *Let H be a closed subgroup of $GL_n(\mathbb{R})$ and let $G = \mathbb{R}^n \rtimes H$. Let $\hat{\mathcal{O}}_{\vec{k}_j^T}$ be an open free H-orbit in $\hat{\mathbb{R}}^n$. Then the restriction $\hat{U}_j(\vec{b}, \mathbf{h})$, of the quasiregular representation to the Hilbert space $L^2(\hat{\mathcal{O}}, \mathrm{d}\vec{k}^T)$, is irreducible and square integrable. The corresponding Duflo–Moore operator C_j assumes the form:*

$$(C_j f)(\vec{k}^T) = (2\pi)^{\frac{n}{2}}[c_j(\vec{k}^T)]^{\frac{1}{2}} f(\vec{k}^T) \tag{61}$$

on $L^2\big(\hat{\mathcal{O}}_{\vec{k}_j^T}, \mathrm{d}\vec{k}^T\big)$, where $c_j : \hat{\mathcal{O}}_{\vec{k}_j^T} \longrightarrow \mathbb{R}^+$ is a positive, Lebesgue measurable function which transforms under the action of H as

$$c_j(\vec{k}^T\mathbf{h}) = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|} c_j(\vec{k}^T) \tag{62}$$

for almost all $\vec{k}^T$ (with respect to the Lebesgue measure). Furthermore, every irreducible representation of G is of this type and the quasi-regular representation is a multiplicity-free direct sum of these representations.

It has also been shown in [11] that $c_j(\vec{k}^T)$ is precisely the density function which converts the Lebesgue measure $\mathrm{d}\vec{k}^T$, restricted to the orbit $\hat{\mathcal{O}}_{\vec{k}_j^T}$, to the invariant measure $\mathrm{d}\nu_j$ on it:

$$\mathrm{d}\nu_j(\vec{k}^T) = c_j(\vec{k}^T)\, \mathrm{d}\vec{k}^T \qquad \text{and} \qquad \mathrm{d}\nu_j(\vec{k}^T\mathbf{h}) = \mathrm{d}\nu_j(\vec{k}^T) \tag{63}$$

and can be defined simply to be the transform of the left Haar measure $\mathrm{d}\mu_H$ of H under the homeomorphism of equation (43),

$$\mathrm{d}\nu_j(\vec{k}^T) = \mathrm{d}\mu_H(\kappa_j(\vec{k}^T)). \tag{64}$$

If $\vec{\gamma}_p^T \in \hat{\mathcal{O}}_{\vec{k}_j^T}$ is an arbitrary point and $\vec{\gamma}_p^T = \vec{k}_j^T\mathbf{h}^{-1}$, (see equation (43)), then in view of equation (62) we may set

$$c_j\big(\vec{\gamma}_p^T\big) = \lambda\frac{|\det \mathbf{h}|}{\Delta_H(\mathbf{h})}$$

for almost all $\vec{\gamma}_p^T \in \hat{\mathcal{O}}_{\vec{k}_j^T}$ (with respect to the Lebesgue measure), where λ is a constant. (Clearly, with this choice of of the density $c_j\big(\vec{\gamma}_p^T\big)$ the invariance condition in equation (63) is satisfied.) In view of equation (64) we may, by multiplying $\mathrm{d}\mu_H$ by a constant if necessary, make $\lambda = 1$. Assuming that this has been done, we may write (for almost all $\vec{\gamma}_p^T$),

$$c_j\big(\vec{\gamma}_p^T\big) = \frac{\big|\det\big[\kappa_j\big(\vec{\gamma}_p^T\big)\big]\big|}{\Delta_H\big[\kappa_j\big(\vec{\gamma}_p^T\big)\big]} \tag{65}$$

Comparing with equations (10), (44) and (54), and using the homeomorphism $\tilde{\kappa}_j : \mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)} \to \mathbb{R}^n \rtimes H$ in (45), we have the result

Theorem 6.2. *Let H be a closed subgroup of $GL_n(\mathbb{R})$ and let $G = \mathbb{R}^n \rtimes H$. Let $\hat{\mathcal{O}}_{\vec{k}_j^T}$ be an open free H-orbit in $\hat{\mathbb{R}}^n$ and let $T^*\hat{\mathcal{O}}_{\vec{k}_j^T}$ be its cotangent bundle with invariant measure $\mathrm{d}\Omega_j$. Then the following equalities hold:*

$$c_j\big(\vec{\gamma}_p^T\big)^{-1} = \sigma_j(\vec{\gamma}^T) = (\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}} = \Delta_G[\tilde{\kappa}_j(\vec{\gamma}^T)] \tag{66}$$

(except at most on a set of measure zero), where c_j is the function defining the Duflo–Moore operator of the unitary irreducible representation $\hat{U}_j$ of G, associated to the orbit $\hat{\mathcal{O}}_{\vec{k}_j^T}$, $\sigma_j(\vec{\gamma}^T)$ is the Radon–Nikodym derivative, $\mathrm{d}\vec{\gamma}^T/\mathrm{d}\Omega_j$, at the point $\vec{\gamma}^T = \big(\vec{\gamma}_q^T, \vec{\gamma}_p^T\big) = (\gamma^1, \gamma^2, \ldots, \gamma^n, \gamma^{n+1}, \ldots, \gamma^{2n}) \in T^*\hat{\mathcal{O}}_{\vec{k}_j^T}$, $[\Theta(\vec{\gamma}^T)]^{ij} = \sum_{k=1}^{2n} c_k^{ij}\gamma^k$, c_k^{ij} being the structure constants of G, and $\tilde{\kappa}_j$ the homeomorphism between $T^*\hat{\mathcal{O}}_{\vec{k}_j^T}$ and $\mathbb{R}^n \rtimes H$, normalized so that $\tilde{\kappa}_j\big(\vec{0}^T, \vec{k}_j^T\big) = (\vec{0}, e)$.

7. Construction of general Wigner functions

It is known (see, for example [9]) that the orthogonality relations of equation (60) for a square integrable representation U of the group G have an extension to Hilbert–Schmidt operators on $\mathfrak{H}$. Let $\mathcal{B}_2(\mathfrak{H})$ denote the Hilbert space of all Hilbert–Schmidt operators ρ on $\mathfrak{H}$. This Hilbert space is equipped with the scalar product

$$\langle \rho_1 \mid \rho_2 \rangle_{\mathcal{B}} = \mathrm{Tr}[\rho_1^*\rho_2].$$

Then there exists a dense set $\mathcal{D} \subset \mathcal{B}_2(\mathfrak{H})$ such that for any $\rho \in \mathcal{D}$, the (closure of) the operator $U(g)^*\rho C^{-1}$ is of trace class (C being the Duflo–Moore operator). Furthermore, the function

$$f_\rho(g) = \mathrm{Tr}[U(g)^*\rho C^{-1}] \tag{67}$$

is an element of $L^2(G, \mathrm{d}\mu_G)$ and moreover,

$$\|f_\rho\|^2_{L^2(G,\mathrm{d}\mu_G)} = \|\rho\|^2_{\mathcal{B}}. \tag{68}$$

Thus, we may define an isometric linear map, $\tilde{\mathfrak{W}} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(G, \mathrm{d}\mu_G)$ which, for $\rho \in \mathcal{D}$, is given by equation (67) and is then extended by continuity to all of $\mathcal{B}_2(\mathfrak{H})$.

We are now ready to give the definition of the general Wigner function. However, it is first necessary to make an additional assumption on the group G, that there exists a symmetric subset N_0 of Lie algebra $\mathfrak{g}$, such that the exponential map restricted to it is a bijection onto a dense set (in G), the complement of which has Haar measure zero: $\mu(G - \exp(N_0)) = 0$. The Wigner function is then defined as a Fourier-like transform of equation (67):

$$W(\rho|\vec{\gamma}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} \mathrm{e}^{-\mathrm{i}\vec{\gamma}^T\vec{x}}\, \mathrm{Tr}[U(\mathrm{e}^{-X}\rho C^{-1}][\sigma(\vec{\gamma}^T)m_G(\vec{x})]^{\frac{1}{2}}\, \mathrm{d}\vec{x} \tag{69}$$

or equivalently, if $\rho = |\phi\rangle\langle\psi|$,

$$W(\phi, \psi|\vec{\gamma}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} \mathrm{e}^{-\mathrm{i}\vec{\gamma}^T\vec{x}} \langle C^{-1}\psi|U(\mathrm{e}^{-X})\phi\rangle[\sigma(\vec{\gamma}^T)m_G(\vec{x})]^{\frac{1}{2}}\, \mathrm{d}\vec{x}. \tag{70}$$

In this expression, $\vec{\gamma}^T \in \mathfrak{g}$, $\phi \in \mathfrak{H}$ and ψ is in the range of the Duflo–Moore operator C related to the representation U. The density function m_G again expresses the Haar measure on G in terms of the Lebesgue measure $\mathrm{d}\vec{x}$ on $\mathfrak{g}$ (see equation (26)):

$$\mathrm{d}\mu_G(\mathrm{e}^X) = m_G(\vec{x})\,\mathrm{d}\vec{x} \tag{71}$$

where $X \in \mathfrak{g}$ is expressed in terms of the components of the vector $\vec{x}$ in the basis $\{L^1, L^2, \ldots, L^{2n}\}$ (i.e., $X = \sum_{i=1}^{2n} x_i L^i$). The function σ is defined by expressing the Lebesgue measure $\mathrm{d}X^*$ in $\mathfrak{g}^*$ in terms of invariant measures $\mathrm{d}\Omega_\lambda$ on the coadjoint orbits $\mathcal{O}^*_\lambda$ as follows:

$$\mathrm{d}X^* = \mathrm{d}\kappa(\lambda)\sigma_\lambda(X^*)\,\mathrm{d}\Omega_\lambda(X^*) \tag{72}$$

where the index λ parametrizes coadjoint orbit and $\mathrm{d}\kappa(\lambda)$ is a measure on the parameter space. This decomposition is not guaranteed in general and has to be assumed or proved for specific cases.

8. Basic properties of general Wigner functions

The appearance of σ_λ in the formula for the Wigner function is necessary in order to have the following important covariance property:

$$W(U(g)\rho U(g)^*|\vec{\gamma}^T) = W(\rho|Ad_{g^{-1}}{}^{\#}\vec{\gamma}^T) \tag{73}$$

which clearly can be regarded as a generalization of the covariance in equation (4) for the original Wigner function. The overlap condition of equation (7) in this more general setting becomes

$$\int_{\mathfrak{g}^*} \overline{W(\rho_1|\vec{\gamma}^T)}W(\rho_2|\vec{\gamma}^T)[\sigma(\vec{\gamma})^T]^{-1}\,\mathrm{d}\vec{\gamma}^T = \mathrm{Tr}[\rho_1^*\rho_2]. \tag{74}$$

As expected, the general Wigner function does not enjoy all the properties of the original function W^{QM}. For example, not both marginal properties of equations (5) and (6) are generally satisfied, or can be given a natural meaning, in view of their dependence on a special choice of coordinates for the phase space over which the function is defined. Additionally, the domain of the general Wigner function could span more than a single coadjoint orbit, and hence more than a single physical phase space. We shall return to these points later, in the context of semidirect product groups of the type described in section 3.

9. General Wigner functions for semidirect product groups

Before proceeding to the case of semidirect product groups, a few comments about the general construction are in order. The original Wigner function, presented in section 2 has the same form as the general Wigner function, with both density functions $m(\vec{x})$ and $\sigma(\vec{\gamma}^T)$ being equal to one and the Duflo–Moore operator being a multiple of identity operator. There is a more subtle difference, however: the original Wigner function arises from an irreducible representation of the Heisenberg–Weyl group G_{HW}, and these representations are only square integrable with respect to the homogeneous space G_{HW}/Θ (Θ being the phase group), and not with respect to the whole group. (On the other hand, as shown in [7], this additional generality is only an apparent one, in this case, and is easily subsumed in a more general theory, based on the Plancherel transform.) It is nevertheless still worth stressing here, that the general procedure for constructing Wigner functions presented in this paper does rest on two requirements: first, the decomposability in equation (72) of the Lebesgue measure $\mathrm{d}X^*$ and secondly, that we consider only groups with square integrable representations.

Semidirect product groups, with open free orbits, which we consider here, satisfy both these conditions. Using their square integrable representations (see equation (58)) we can rewrite the general Wigner function in equation (57) as

$$W(\hat{\phi},\hat{\psi}|\vec{\gamma}^T)=\frac{1}{(2\pi)^n}\int_{N_0}\mathrm{d}\vec{x}\,\mathrm{e}^{-\mathrm{i}\vec{\gamma}^T\vec{x}}\int_{\mathcal{O}^*}\mathrm{d}\vec{\omega}^T\overline{C^{-1}\hat{\psi}(\vec{\omega}^T)}|\det\mathrm{e}^{-X_q}|^{\frac{1}{2}}$$
$$\times\exp\left(\mathrm{i}\vec{\omega}^T\mathrm{e}^{-\frac{X_q}{2}}\operatorname{sinch}\frac{X_q}{2}\vec{x}_p\right)\hat{\phi}(\vec{\omega}^T\mathrm{e}^{-X_q})[\sigma(\vec{\gamma}^T)m(\vec{x})]^{\frac{1}{2}}. \tag{75}$$

Changing variables: $\vec{\omega}'^T=\vec{\omega}^T\,\mathrm{e}^{-\frac{X_q}{2}}\operatorname{sinch}\frac{X_q}{2}$ and using the form for the density function $m(\vec{x})$ given in equation (31) we obtain:

$$W(\hat{\phi},\hat{\psi}|\vec{\gamma}^T)=\frac{1}{(2\pi)^n}\int_{N_{0q}}\int_{\mathbb{R}^2}\mathrm{d}\vec{x}_q\,\mathrm{d}\vec{x}_p\,\mathrm{e}^{-\mathrm{i}\vec{\gamma}_q^T\vec{x}_q}$$
$$\times\int_{\mathcal{O}^*}\mathrm{d}\vec{\omega}'^T\,\mathrm{e}^{\mathrm{i}(\vec{\omega}'^T-\vec{\gamma}_p^T)\vec{x}_p}\overline{C^{-1}\hat{\psi}\left(\vec{\omega}'^T\frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\omega}'\frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)$$
$$\times\sigma(\vec{\gamma}^T)^{\frac{1}{2}}\left|\det\frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right|^{\frac{1}{2}}\left|\det\left(\mathrm{e}^{-a\,\mathrm{d}\frac{X_q}{2}}\operatorname{sinch}a\,\mathrm{d}\frac{X_q}{2}\right)\right|^{\frac{1}{2}}. \tag{76}$$

We have shown in theorem 6.1 that the Duflo–Moore operator in this case is related to the decomposition (see equation (72)) of the Lebesgue measure in $\mathfrak{g}$ and is expressible in terms of the structure constants of the Lie algebra $\mathfrak{g}$. Applying equation (66) together with equation (62) and integrating over $\vec{x}_p$ we finally obtain:

$$W(\hat{\phi},\hat{\psi}|\vec{\gamma}^T)=\int_{N_{0q}}\mathrm{d}\vec{x}_q\,\mathrm{e}^{-\mathrm{i}\vec{\gamma}_q^T\vec{x}_q}\overline{\hat{\psi}\left(\vec{\gamma}_p^T\frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\gamma}_p\frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)$$
$$\times c\left(\vec{\gamma}_p^T\frac{1}{\operatorname{sinch}\frac{X_q}{2}}\right)^{-\frac{1}{2}}c\left(\vec{\gamma}_p^T\right)^{-\frac{1}{2}}\left|\frac{\det\left(\operatorname{sinch}a\,\mathrm{d}\frac{X_q}{2}\right)}{\det\left(\operatorname{sinch}\frac{X_q}{2}\right)}\right|^{\frac{1}{2}}. \tag{77}$$

Here we used the fact that the domain N_0 of the exponential map $\exp\colon\mathfrak{g}\to G$ in the case of semidirect product groups, is given by $N_{0q}\times\mathbb{R}^n$, where N_{0q} is the corresponding domain the exponential map $\exp\colon\mathfrak{h}\to H$. Again we assume that this map is a bijection onto a dense set in H such that its complement has measure zero.

It is easily seen that if we integrate the Wigner function in equation (77) with respect to the measure $c\left(\vec{\gamma}_p^T\right)\mathrm{d}\vec{\gamma}_q^T$, we obtain a marginality property of the form:

$$\int_{\mathbb{R}^n}W(\hat{\phi},\hat{\psi}|\vec{\gamma}^T)c\left(\vec{\gamma}_p^T\right)\mathrm{d}\gamma_q^T=\overline{\hat{\phi}\left(\vec{\gamma}_p^T\right)}\hat{\psi}\left(\vec{\gamma}_p^T\right) \tag{78}$$

as a generalization of equation (5). Unfortunately, the second property (see equation (6)) does not have a simple form any longer.

10. Domain of the Wigner function

As mentioned earlier, the advantage of using the original Wigner function is that it allows us to represent a quantum state or a signal (in the case of signal analysis) as a function on a phase space (position-momentum or time-frequency). Thus, ideally, we would like our Wigner function to be supported on a single coadjoint orbit (which together with its symplectic form ω_{X^*} in equation (52) can be considered as a phase space). We investigate now, under what conditions this is the case for the Wigner functions just derived.

Recall first that the open free coadjoint orbits $\mathcal{O}_i^*$ in $\mathfrak{g}^*$, for semidirect product groups, are in one-to-one correspondence with open free H-orbits $\hat{\mathcal{O}}_i \subset \hat{\mathbb{R}}^n$: indeed according to theorem 4.1, any $\mathcal{O}_i^*$ is a cotangent bundle of the form $\mathcal{O}_i^* = T^*\hat{\mathcal{O}}_i = \hat{\mathcal{O}}_i \times \mathbb{R}^n$. Let $W_{\hat{\mathcal{O}}_i}$ denote the Wigner function derived from a representation of G acting on $\mathfrak{H} = L^2(\hat{\mathcal{O}}_i)$, which can be conveniently thought of as the closed subspace of $L^2(\hat{\mathbb{R}}^n, \mathrm{d}\vec{x})$ of functions which vanish almost everywhere outside $\hat{\mathcal{O}}_i$. We are going to find sufficient conditions for the Wigner function $W_{\hat{\mathcal{O}}_i}$ to have support concentrated on the corresponding coadjoint orbit $\mathcal{O}_i^* = \hat{\mathcal{O}}_i \times \mathbb{R}^n$.

Let us start by introducing a polynomial function Δ,

$$\Delta(\vec{\gamma}^T) = \det \begin{pmatrix} \vec{\gamma}^T L_1 \\ \vec{\gamma}^T L_2 \\ \vdots \\ \vec{\gamma}^T L_n \end{pmatrix} \tag{79}$$

where $\{L_1, L_2, \ldots, L_n\}$ is a basis in $\mathfrak{g}$. It is very convenient to discuss the orbit structure in $\hat{\mathbb{R}}^n$ with the use of this function Δ. A point $\vec{\gamma}^T$ belongs to the open (free) orbit if and only if it satisfies $\Delta(x) \neq 0$ and a point belongs to an orbit of dimension $<n$ if and only if $\Delta(\vec{\gamma}^T) = 0$. We have the following:

Proposition 10.1. *Let G be a semidirect product group $\mathbb{R}^n \rtimes H$, such that H acts on $\hat{\mathbb{R}}^n$ with open, free orbits $\{\hat{\mathcal{O}}_i\}_{i=1}^m$. If an orbit $\hat{\mathcal{O}}_i$ is a dihedral cone (i.e. if the zero level set of the function Δ in equation (79), restricted to it, can be decomposed into hyperplanes) then the Wigner function $W_{\hat{\mathcal{O}}_i}$ has support concentrated on the corresponding coadjoint orbit $\mathcal{O}_i^* = \mathbb{R}^n \times \hat{\mathcal{O}}_i$.*

To prove it we will need the following lemma:

Lemma 10.2. *If a hyperplane $\Pi(\vec{\gamma}^T) = 0$ is a subset of $\Delta(\vec{\gamma}^T) = 0$ then it is invariant under H.*

Proof of lemma 10.2. We will show first that we can always find $\vec{\gamma}_0^T \in \Pi^{-1}(0)$ such that there exists a neighbourhood $U_{\vec{\gamma}_0^T}$ satisfying

$$U_{\vec{\gamma}_0^T} \cap \Pi^{-1}(0) = U_{\vec{\gamma}_0^T} \cap \Delta^{-1}(0). \tag{80}$$

Let us introduce a basis $\{Z_1, \ldots, Z_n\}$ in $\hat{\mathbb{R}}^n$ such that the first $n-1$ elements constitute a basis in the hyperplane $\Pi(\vec{\gamma}^T) = 0$. In these coordinates, the the hyperplane can be written as $\Pi(\vec{\gamma}^T) = \gamma^n$ and the function Δ can be factored as

$$\Delta(\gamma^T) = (\gamma^n)^k P(\vec{\gamma}^T) \tag{81}$$

such that $P(\vec{\gamma})$ does not contain γ^n as a factor. Then $P(\gamma^1, \ldots, \gamma^{n-1}, 0) \equiv 0$ iff $P(\vec{\gamma}) \equiv 0$ (which would imply $\Delta(\vec{\gamma}) \equiv 0$, a contradiction) or $P(\vec{\gamma})$ contains γ^n as a factor, which would contradict equation (81). Thus we can always choose $\vec{\gamma}_0 = (\gamma_1, \ldots, \gamma_{n-1}, 0)$ such that $P(\vec{\gamma}_0) = r \neq 0$ and $\Pi(\vec{\gamma}_0) = 0$. Since P is a polynomial, there exists an open neighbourhood $U_{\vec{\gamma}_0^T}$ of γ_0^T such that $P\left(\vec{\gamma}_0^T\right) \in (r-\epsilon, r+\epsilon)$. Thus equation (80) is true.

Now, for any $\vec{\gamma}^T \in U_{\vec{\gamma}_0^T}$ the intersection of its orbit $\mathcal{O}_{\vec{\gamma}^T}$ with $U_{\vec{\gamma}_0^T}$ belongs to Π. This implies that for every $\vec{\gamma}^T \in U_{\vec{\gamma}_0^T}$, $\mathfrak{h}\vec{\gamma}^T \in \Pi$. We can choose a basis of Π formed by $N-1$ linearly independent elements $\{Z_1', \ldots, Z_{N-1}'\} \subset U_{\vec{\gamma}_0^T}$. Since $\mathfrak{h}Z_i' \subset \Pi$ is true for every basis element, then also, for every $\vec{\gamma}^T \in \Pi$, $\mathfrak{h}\vec{\gamma}^T \subset \Pi$, i.e. the hyperplane Π is stable under $\mathfrak{h}$ and hence also under H (by exponentiation). □

Proof of proposition 10.1. One sees from equation (77) that a sufficient condition for the Wigner transform to preserve the decomposition into orbits $\mathcal{O}_i^*$ is that the point $\vec{\gamma}^T \frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}$ does not leave $\hat{\mathcal{O}}_i$ as X_q varies in N_{0q}, or equivalently, that the 'sinch' map preserve orbits (which is not guaranteed because sinch (X) is not an element of the group H). Let us take again a basis $\{Z_1, \ldots, Z_{n-1}, Z_n\}$ in $\hat{\mathbb{R}}^n$ such as the first $n-1$ elements belong to the $(n-1)$-dim hyperplane as in lemma 10.2. In the coordinates introduced above, an element X of the Lie algebra $\mathfrak{h}$ of the group H is of the form:

$$X = \begin{pmatrix} X_{1,1} & \ldots & X_{1,n-1} & 0 \\ \ldots & & & \\ X_{n-1,1} & \ldots & X_{n-1,n-1} & 0 \\ X_{n,1} & \ldots & X_{n,n-1} & X_{n,n} \end{pmatrix}$$

because X must preserve the hyperplane $\gamma^n = 0$. Calculating the sinch of such element X we obtain

$$S = \operatorname{sinch}(X) = \begin{pmatrix} S_{1,1} & \ldots & S_{1,n-1} & 0 \\ \ldots & & & \\ S_{n-1,1} & \ldots & S_{n-1,n-1} & 0 \\ S_{n,1} & \ldots & S_{n,n-1} & \operatorname{sinch}(X_{n,n}) \end{pmatrix}.$$

Note that $\operatorname{sinch}(X_{n,n}) > 0$ from definition in equation (22). Applying $\operatorname{sinch}(X)$ to any vector $\vec{\gamma}^T$ in $\hat{\mathbb{R}}^n$, written in the basis $\{Z_i\}_1^n$, we have

$$\operatorname{sinch}(X)(\gamma^1, \ldots, \gamma^{n-1}, \gamma^n) = (\gamma^{1'}, \ldots, \gamma^{n-1'}, \operatorname{sinch}(X_{n,n}))\gamma^n).$$

Therefore, the sign of γ^n remains unchanged, which also means that the hyperplane $\gamma^n = 0$ divides $\hat{\mathbb{R}}^n$ into two half spaces, invariant under the sinch map. Since $\Delta^{-1}(0)$ is a union of hyperplanes $\Pi_1 \cup \Pi_2 \ldots \cup \Pi_r$, we can repeat the argument for each of them, proving that each open orbit is preserved. □

11. Examples

It is now an easy task to explicitly compute Wigner functions for particular cases of groups from its general form for semidirect product groups in equation (77). Let us consider first examples of connected four-dimensional semidirect product groups $G = \mathbb{R}^2 \rtimes H$ with open free H-orbits in $\hat{\mathbb{R}}^2$, i.e. when H is diagonal group, SIM(2) or one of the infinite family of H_c groups. We work out the Wigner functions for all such groups. Next we present an interesting example of an eight-dimensional group $G = H \rtimes H^*$, where H is a vector space of quaternions and H^* a group of invertible quaternions. The Wigner functions constructed here are all candidates for use in image analysis in various dimensions (see also [2], [6] and [12]).

11.1. The diagonal group

Let $G = \mathbb{R}^2 \rtimes H$ where H is the diagonal subgroup of $GL_2(\mathbb{R})$ that is

$$H = \operatorname{diag}(a_1, a_2) \qquad a_1, a_2 \in \mathbb{R} - \{0\}.$$

The Wigner functions are defined on the coadjoint G-orbits, $\mathcal{O}^*_{\vec{\gamma}^T_{ij}}$, of the elements $\vec{\gamma}^T_{ij} = (0, 0, i, j)$, $i = \pm 1$, $j = \pm 1$, the union of which is dense in $\hat{\mathbb{R}}^4$.

$$W(\hat{\phi}, \hat{\psi}|\vec{\gamma}^T) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{e}^{-\mathrm{i}\gamma^1 x_1 - \mathrm{i}\gamma^2 x_2}\overline{\hat{\psi}\left(\gamma_1 \frac{\mathrm{e}^{\frac{x_1}{2}}}{\operatorname{sinch}\frac{x_1}{2}}\right)}$$
$$\times\,\hat{\phi}\left(\gamma_2\frac{\mathrm{e}^{-\frac{x_2}{2}}}{\operatorname{sinch}\frac{x_2}{2}}\right)\frac{|\gamma^3\gamma^4|}{\operatorname{sinch}\frac{x_1}{2}\operatorname{sinch}\frac{x_2}{2}}, \tag{82}$$

where an element of $\mathfrak{h}$ has the form $X_q = \mathrm{diag}(x_1, x_2)$ and the corresponding element in H is $\mathrm{e}^{X_q} = \mathrm{diag}(\mathrm{e}^{x_1}, \mathrm{e}^{x_2})$. We have also used the following relations:

$$\operatorname{sinch}\frac{X_q}{2} = \mathrm{diag}\left(\operatorname{sinch}\frac{x_1}{2}, \operatorname{sinch}\frac{x_2}{2}\right) \qquad c\left(\vec{\gamma}_p^T\right) = |\gamma^3\gamma^4|.$$

11.2. The SIM(2) group

Let G denote the $SIM(2)$ group, i.e., the group of dilations, rotations and translations in $\mathbb{R}^2$. $G = \mathbb{R}^2 \rtimes H$ where $H = \left\{\begin{pmatrix} a & -b \\ b & a\end{pmatrix} : (a, b) \in \mathbb{R}^2 - \{0, 0\}\right\}$. The Wigner function is defined on $\mathcal{O}^*_{\vec{\gamma}_0^T} = \{(\gamma^1, \gamma^2, \gamma^3, \gamma^4) : (\gamma_3, \gamma_4) \neq (0, 0)\}$ (the coadjoint orbit of $\vec{\gamma}_0 = (0, 0, 1, 0)$). This case was studied extensively in [6]. This time, however, we can make use of the Wigner function for semidirect product groups to obtain the result immediately.

Denoting the element of the lie algebra $\mathfrak{h}$ as $X_q = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda\end{pmatrix}$, $\theta \in (0, 2\pi)$, $\lambda \geqslant 0$ and the corresponding element of the group H as $\mathrm{e}^{X_q} = \begin{pmatrix} \mathrm{e}^\lambda\cos\theta & -\mathrm{e}^\lambda\sin\theta \\ \mathrm{e}^\lambda\sin\theta & \mathrm{e}^\lambda\cos\theta\end{pmatrix}$ the corresponding Wigner function is

$$W(\hat{\phi}, \hat{\psi}|\vec{\gamma}^T) = \frac{(\gamma^3)^2 + (\gamma^4)^2}{2\pi}\int_{N_0 q} \mathrm{e}^{-\mathrm{i}\gamma_1\lambda - \mathrm{i}\gamma_2\theta}\overline{\hat{\psi}\left(\vec{\gamma}_p\frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\gamma}_p\frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)$$
$$\times\,\frac{\lambda^2 + \theta^2}{2\cosh\lambda - 2\cos\theta}\,\mathrm{d}\lambda\,\mathrm{d}\theta. \tag{83}$$

We have used the following relations:

$$\det\left[\operatorname{sinch}\frac{X_q}{2}\right] = \frac{2\cosh\lambda - 2\cos\theta}{\lambda^2 + \theta^2} \qquad \text{and} \qquad c\left(\vec{\gamma}_p^T\right) = |(\gamma^3)^2 + (\gamma^4)^2|^{-1}.$$

Since H is Abelian, $\det\left[\operatorname{sinch} a\,\mathrm{d}\frac{X_q}{2}\right] = 1$.

11.3. The one-parameter family of groups H_c

Consider now the one parameter family of groups $H_c = \left\{\begin{pmatrix} a & 0 \\ b & a^c\end{pmatrix} : a, b \in \mathbb{R}, a > 0\right\}$ for $c \neq 0$. The Wigner functions are defined on coadjoint orbits $\mathcal{O}_+ = \{(\gamma^1, \gamma^2, \gamma^3, \gamma^4 : \gamma^4 > 0\}$ and $\mathcal{O}_- = \{(\gamma^1, \gamma^2, \gamma^3, \gamma^4 : \gamma^4 < 0\}$.

An element of the Lie algebra $\mathfrak{h}_c$ is $X_q = \begin{pmatrix} x_1 & 0 \\ x_2 & cx_1\end{pmatrix}$, and the corresponding element of a group H_c is $\mathrm{e}^{X_q} = \begin{pmatrix} \mathrm{e}^{x_1} & 0 \\ \frac{x_2}{(c-1)x_1}(\mathrm{e}^{x_1 c} - \mathrm{e}^{x_1}) & \mathrm{e}^{x_1 c}\end{pmatrix}$ (in the case c = 1 we should take the $\lim_{c\to 1}$).

The Wigner function takes the form

$$W(\hat{\phi}, \hat{\psi}|\vec{\gamma}^T) = \frac{(|\gamma^4|^2)}{2\pi}\int_{N_0 q} \mathrm{e}^{-\mathrm{i}\gamma^1 x_1 - \mathrm{i}\gamma^2 x_2}\overline{\hat{\psi}\left(\vec{\gamma}_p\frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\gamma}_p\frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)$$
$$\times\,\frac{1}{\operatorname{sinch}\frac{x_1 c}{2}}\left(\frac{\operatorname{sinch}\frac{x_1(c-1)}{2}}{\operatorname{sinch}\frac{x_1}{2}\operatorname{sinch}\frac{x_1 c}{2}}\right)^{\frac{1}{2}}\mathrm{d}x_1\,\mathrm{d}x_2 \tag{84}$$

where we have used the following relations:

$$\operatorname{sinch}\frac{X_q}{2} = \begin{pmatrix} \operatorname{sinch}\frac{x_1}{2} & 0 \\ \frac{1}{1-c}\frac{x_2}{x_1}\left(\operatorname{sinch}\frac{x_1}{2} - \operatorname{sinch}\frac{cx_1}{2}\right) & \operatorname{sinch}\frac{cx_1}{2} \end{pmatrix}$$

$$\det\left[\operatorname{sinch} a\,\mathrm{d}\frac{X_q}{2}\right] = \operatorname{sinch}\frac{(c-1)x_1}{2} \qquad \text{and} \qquad c(\vec{\gamma}_p) = |\gamma^4|^{-2}.$$

11.4. Quaternionic groups

Quaternions constitute a (non-Abelian) field of numbers; they can be thought of as an extension of the complex numbers, similar to the way that complex numbers are an extension of the real numbers. More specifically, they are obtained by adding two more 'imaginary units', customarily denoted by **j** and **k**, such that the following relations are fulfilled:

$$\mathbf{j}^2 = \mathbf{i}^2 = \mathbf{k}^2 = -1 \qquad \mathbf{ij} = \mathbf{k} \qquad \mathbf{jk} = \mathbf{i} \qquad \mathbf{ki} = \mathbf{j}. \tag{85}$$

The generic quaternion can be written as $x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where x_0, x_1, x_2, x_3 are real numbers, or as $z_0 + z_1\mathbf{j}$, where $z_0 = x_0 + x_1\mathbf{i}$ and $z_1 = x_2 + x_3\mathbf{i}$ are complex numbers. A very practical way of dealing with quaternions is to represent them as 2×2 matrices with complex entries

$$q := \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}. \tag{86}$$

By identifying the set of quaternions with $\mathbb{R}^4$ one can endow this latter space with a notion of multiplication. It is also worthwhile to recall that any nonzero quaternion admits a (multiplicative) inverse which can be expressed by taking the inverse of the matrix representing it.

Let us consider now the semidirect product group $G = H \rtimes H^*$ where H denotes the vector space of quaternions and H^* the group of invertible quaternions. An element of the group can be written in the form:

$$g = \begin{pmatrix} h_q & h_p \\ 0 & 1 \end{pmatrix}$$

where $h_q \in H^*$ and $h_p \in H$. Then an element of the lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is:

$$X = \begin{pmatrix} X_q & X_p \\ 0 & 0 \end{pmatrix}$$

where X_q and X_p are both quaternions which can be written in coordinates as:

$$X_q = \begin{pmatrix} x_0 + \mathrm{i}x_1 & x_2 + \mathrm{i}x_3 \\ -x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix} \qquad X_p = \begin{pmatrix} x_4 + \mathrm{i}x_5 & x_6 + \mathrm{i}x_7 \\ -x_6 + \mathrm{i}x_7 & x_4 - \mathrm{i}x_5 \end{pmatrix}.$$

The group can be equivalently written, in a manner more consistent with the rest of this paper, as $\mathbb{R}^4 \rtimes M(h_q)$ where $M(h_q) \in GL(4, \mathbb{R})$ is of the form

$$M(h_q) = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}.$$

The quaternionic notation makes it easy to relate this group to the $G_1 = \mathbb{R} \rtimes \mathbb{R}^*$ and $G_2 = \mathbb{C} \rtimes \mathbb{C}^* = \mathit{SIM}(2)$, which are the wavelet groups in one and two dimensions, respectively. It seems quite natural to use the field of quaternions to define a wavelet group in four dimensions. (The concept of wavelet groups can therefore be extended in a rather straightforward way to any Clifford algebra.) The Wigner function is defined on the single coadjoint orbit $\mathcal{O}^* = \mathfrak{g}^* - \{0\}$. It has the form

$$W(\hat{\phi}, \hat{\psi}|\vec{X}^*) = \frac{|X_p^*|^4}{(2\pi)^4} \int_{\mathbb{R}^4} \mathrm{e}^{-\mathrm{i}\langle X_q^*, X_q\rangle} \overline{\hat{\psi}\left(X_p^* \frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}}\right)} \hat{\phi}\left(X_p^* \frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}}\right) \times \frac{1}{16} \frac{|X_q|^4}{\left(\cosh^2 \frac{x_0}{2} - \cos^2 \frac{R}{2}\right)^2} \frac{\sin R}{R} \, \mathrm{d}X_q \tag{87}$$

where $R = \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{2}}$. We have also used

$$c(X^*) = |X_p^*|^{-4} \qquad \text{and} \qquad \mathrm{d}\mu_G(\mathrm{e}^X) = \det\left(\mathrm{e}^{-\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2}\right) \frac{\sin R}{R} \, \mathrm{d}X.$$

All these computations can be easily repeated for any Clifford algebra.

11.5. A group H which does not satisfy the assumption of theorem 10.1

Consider a three-dimensional group H, the Lie algebra $\mathfrak{h}$ of which is generated by the following elements:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; F^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{88}$$

The orbit structure in $\hat{\mathbb{R}}^3$ is given by the equation: $\Delta(\vec{\omega}^T) = -\frac{1}{3}\omega_3\left(-2\omega_3\omega_1 + \omega_2^2\right) = 0$, which clearly cannot be decomposed into hyperplanes. We have the following open orbits in $\hat{\mathbb{R}}^n$:

$\hat{\mathcal{O}}_1$—above the hyperplane $\omega_3 = 0$ and inside the cone $-2\omega_3\omega_1 + \omega_2^2 < 0 (\Delta > 0)$,
$\hat{\mathcal{O}}_2$—above the hyperplane $\omega_3 = 0$ and outside the cone $-2\omega_3\omega_1 + \omega_2^2 > 0 (\Delta < 0)$,
$\hat{\mathcal{O}}_3$—below the hyperplane $\omega_3 = 0$ and inside the cone $-2\omega_3\omega_1 + \omega_2^2 < 0 (\Delta > 0)$,
$\hat{\mathcal{O}}_4$—below the hyperplane $\omega_3 = 0$ and outside the cone $\left(-2\omega_3\omega_1 + \omega_2^2 > 0 \Delta < 0\right)$.

In order to see that the sinch map does not preserve orbits let us choose a point in $\hat{\mathcal{O}}_2$: $\vec{\omega}_0^T = (\omega_1, \omega_2, \omega_3)$ and apply to it $\operatorname{sinch}(tF^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{6}t^2 & 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$. Then one can compute $\Delta\left(\vec{\omega}_0^T \operatorname{sinch}(tF^1)\right) = \frac{1}{9}\omega_3\left(6\omega_3\omega_1 + {\omega_3}^2 t^2 - 3{\omega_2}^2\right)$. It is clear that, as a function of t, it changes sign whenever $2\omega_3\omega_1 - {\omega_2}^2 < 0$. This also means that the sinch map mixes two orbits, $\hat{\mathcal{O}}_1$ with $\hat{\mathcal{O}}_2$ and also $\hat{\mathcal{O}}_3$ with $\hat{\mathcal{O}}_4$. Using a continuity argument, this mixing property holds for a suitable open neighbourhood of F^1 in the Lie algebra, i.e. a set of positive Lebesgue measure. As a consequence, a Wigner function $W(\hat{\phi}, \hat{\psi}|X^*)$ corresponding to two functions supported in $\hat{\mathcal{O}}_1$, $\hat{\phi}, \hat{\psi} \in L^2(\hat{\mathcal{O}}_1)$, will have its support spread on *both* coadjoint orbits $\mathcal{O}_1^*$ and $\mathcal{O}_2^*$. To see that let us fix $\vec{\gamma}_p^T = \vec{\omega}_0^T \in \hat{\mathcal{O}}_2$. Then the Wigner function, as a function of $\vec{\gamma}_q^T \in \hat{\mathbb{R}}^n$, is just the Fourier transform of a function $F(X_q)$

$$W_{\omega_0^T}\left(\hat{\phi}, \hat{\psi}|\vec{\gamma}_q^T\right) = \int_{\mathbb{R}^n} \mathrm{d}\vec{x}_q \, \mathrm{e}^{-\mathrm{i}\vec{\gamma}_q^T \vec{x}_q} F(X_q) \tag{89}$$

where

$$F(X_q) = \overline{\hat{\psi}\left(\vec{\omega}_0^T \frac{\mathrm{e}^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\omega}_0^T \frac{\mathrm{e}^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}\right)$$
$$\times c\left(\vec{\omega}_0^T \frac{1}{\operatorname{sinch}\frac{X_q}{2}}\right)^{-\frac{1}{2}} c\left(\vec{\omega}_0^T\right)^{-\frac{1}{2}} \left|\frac{\det\left(\operatorname{sinch} a\,\mathrm{d}\frac{X_q}{2}\right)}{\det\left(\operatorname{sinch}\frac{X_q}{2}\right)}\right|^{\frac{1}{2}}. \tag{90}$$

Since the map $\operatorname{sinch}(X_q)$ (and $\operatorname{sinch}(X_q)^{-1}$ by the same argument) brings $\vec{\omega}_0^T$ from $\hat{\mathcal{O}}_2$ to $\hat{\mathcal{O}}_1$ (support of $\hat{\phi}$, $\hat{\psi}$) the function $F(X_q)$ is not identically zero, e.g., for X_q in a suitable open neighbourhood of F^1. Hence its Fourier transform $W_{\omega_0^T}(\hat{\phi}, \hat{\psi}|\vec{\gamma}_q{}^T)$ is also not identically zero. This means that the Wigner function $W(\hat{\phi}, \hat{\psi}|X^*)$ does not vanish outside the orbit $\mathcal{O}_1^*$.

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